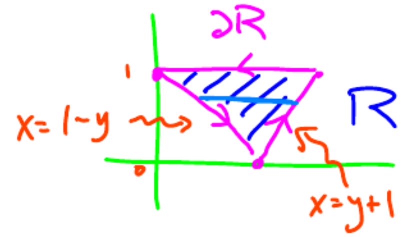
SELECT SOLUTIONS (Green's Theorem)

(1b) Evaluate $\int_{\partial R} (x^2+y^2)\,dx + (x^2-y^2)\,dy$ for R the triangle with vertizes $(0,1)$, $(2,1)$, and $(0,1)$ via Green's Theorem

Sol: Recall Green's Theorem: $\int_{\partial R} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$.

We can parametrize

Pro-tip: Picture!



$R = \{(x,y) : 0 \le y \le 1,\ 1-y \le x \le y+1\}$

Thus we may compute the line integral via:

$$\int_{\partial R} (x^2+y^2)\,dx + (x^2-y^2)\,dy$$

$$= \iint_R \left(\frac{\partial}{\partial x}[x^2-y^2] - \frac{\partial}{\partial y}[x^2+y^2]\right) dA$$

$$= \int_{y=0}^{1} \int_{x=1-y}^{y+1} (2x - 2y)\,dx\,dy$$

$$= \int_{y=0}^{1} \left[x^2 - 2xy\right]_{x=1-y}^{y+1} dy$$

$$= \int_{y=0}^{1} \left(\left((y+1)^2 - 2(y+1)y\right) - \left((1-y)^2 - 2(1-y)y\right)\right) dy$$

$$= \int_{y=0}^{1} \left((y+1+1-y)(y+1-1+y) + 2y(1-y-y-1)\right) dy$$

$$= \int_{y=0}^{1} \left(2(2y) + 2y(-2y)\right) dy = \int_{y=0}^{1} (4y - 4y^2)\,dy$$

$$= \left[2y^2 - \frac{4}{3}y^3\right]_{y=0}^{1} = \left(2 - \frac{4}{3}\right) - (0 - 0) = \frac{2}{3}$$ ▨

(2b) Compute $\int_C \vec{F}\cdot d\vec{r}$ via Green's Theorem for $\vec{F} = \langle e^{-x}+y^2, e^{-y}+x^2\rangle$ where C is the arc of $y=\cos(x)$ from $(-\frac{\pi}{2},0)$ to $(\frac{\pi}{2},0)$ followed by the segment connecting $(\frac{\pi}{2},0)$ to $(-\frac{\pi}{2},0)$.

Pro-tip: Picture!



clockwise, not widdershins !!

Sol: Note C has NEGATIVE ORIENTATION!

(NB: we will adjust by negating our result).

Letting $\vec{r}(t) = \langle x(t), y(t)\rangle$ and $\vec{F} = \langle P, Q\rangle$:

$R = \left\{(x,y) : \begin{matrix} -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 0 \le y \le \cos(x) \end{matrix}\right\}$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{-\partial R} \vec{F}\cdot d\vec{r} = -\int_{\partial R} \vec{F}\cdot d\vec{r}$$

$$= -\int_{\partial R} \vec{F}\cdot \vec{r}\,'\,dt = -\int_{\partial R} P\,x'(t)\,dt + Q\,y'(t)\,dt$$

$$= -\int_{\partial R} P\,dx + Q\,dy = -\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \quad \text{(Green's Theorem)}$$

$$= -\int_{x=-\frac{\pi}{2}}^{\pi/2} \int_{y=0}^{\cos(x)} \left(\frac{\partial}{\partial x}\left[e^{-y}+x^2\right] - \frac{\partial}{\partial y}\left[e^{-x}+y^2\right]\right) dy\,dx$$

$$= -\int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{y=0}^{\cos(x)} (2x-2y)\,dy\,dx = -\int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[2xy - y^2\right]_{y=0}^{\cos(x)} dx$$

$$= -\int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\left(2x\cos(x) - \cos^2(x)\right) - (0-0)\right) dx$$

$$= \int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\left(\cos(x) - 2x\right) dx$$

Now we'll integrate by parts:

$$\int \cos(x)(\cos(x) - 2x)\,dx \qquad \begin{pmatrix} u_1 = \cos(x) - 2x & dv_1 = \cos(x)\,dx \\ du_1 = (-\sin(x) - 2)\,dx & v_1 = \sin(x) \end{pmatrix}$$

$$= \sin(x)(\cos(x) - 2x) + \int (\sin^2(x) + 2\sin(x))\,dx$$

$$= \sin(x)\cos(x) - 2x\sin(x) - 2\cos(x) + \int \tfrac{1}{2}(1 - \cos(2x))\,dx$$

$$= \sin(x)\cos(x) - 2x\sin(x) - 2\cos(x) + \tfrac{1}{2}x - \tfrac{1}{4}\sin(2x) + C$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_{x=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)(\cos(x) - 2x)\,dx$$

$$= \left(0 - 2\cdot\tfrac{\pi}{2} - 0 + \tfrac{1}{2}\cdot\tfrac{\pi}{2} - 0\right) - \left(0 - 2\cdot\tfrac{-\pi}{2} - 0 + \tfrac{1}{2}\cdot\tfrac{-\pi}{2} - 0\right) = -\frac{3\pi}{2} \quad ☐$$